J. Fluid Mech. (1972), *vol.* 55, *part* 3, *pp.* 419–438
Printed in Great Britain

Free and forced convection from fine hot wires

By W. W. WOOD
Department of Mathematics, Melbourne University

(Received 4 July 1969 and in revised form 17 July 1972)

The incipient effect of buoyancy on the heat loss from a fine hot wire in two-dimensional steady incompressible flow is considered. The wire is taken to be horizontal, the mainstream is taken to be normal to the wire and to be directed upwards at an acute angle α to the vertical, and the product $N\epsilon = NG/R^3$ is assumed to be small, where N, G and R denote respectively the Nusselt, Grashof and Reynolds numbers (defined, in the usual way, in §§2 and 3). Three parts of the flow are distinguished and discussed in turn. These are (i) the wire's thermal wake, (ii) an outer irrotational flow induced as a small perturbation of the mainstream by the thermal wake, and (iii) an inner flow, near the wire, in which diffusion either dominates or balances convection. It is shown that the change caused by buoyancy in the wire's heat loss is due largely to the irrotational flow induced by the wire's thermal wake. When $\log N\epsilon$ is significantly large, the change caused by buoyancy in the wire's heat loss is almost entirely due to the irrotational flow. The increment caused by buoyancy in the Nusselt number is then approximately the same as would be produced, in the absence of buoyancy, if the mainstream speed were increased by a factor $1 - 2\sigma^{-1} N\epsilon \log (N\epsilon) \cos \alpha$, where σ is the Prandtl number.

1. Introduction

One aspect of hot-wire anemometry is that the cooling of the hot wire may be affected by the buoyancy that it produces in the passing fluid. Normally, the ambient current is so fast that the buoyancy effect is negligible. But, in slow currents, this may not be so, and inconvenient complications then intrude. For example, the wire's heat loss in mixed (i.e. partly forced and partly natural) convection depends on the angle of the wire to the vertical. Also, the heat loss by a horizontal wire in a horizontal stream in mixed convection does not depend monotonically on the stream speed (Collis & Williams 1959), so that a given heat loss may represent either one of two distinct stream speeds.

For slow currents, then, it is desirable to know fairly precisely when buoyancy can be ignored. We shall consider the case in which the Reynolds number R is not large compared with unity. An inspection of magnitudes indicates that, in this case, the effect of buoyancy on the heat transfer is small if R is much greater than $G^{\frac{1}{3}}$, where G is the Grashof number. This criterion has been confirmed experimentally for air (Collis & Williams 1959), but apparently it needs modification for fluids with large Prandtl numbers (Gebhart & Pera 1971). In air, the condition $R = G^{\frac{1}{3}}$ is achieved by typical hot-wire anemometers in slow draughts with speeds of a few cm s^{-1}.

The aim here is to elucidate the incipient effect of buoyancy on the heat transfer from a wire when $\epsilon = G^{\frac{1}{3}}/R$ is small. For simplicity, the flow is assumed to be steady and two-dimensional and the Prandtl number is assumed to be of $O(1)$. The wire is taken to be horizontal, and normal to the incident stream, and the incident stream is taken to be directed upward at an angle α to the vertical.

To treat the related problem of a hot sphere in a vertical updraught, Hieber & Gebhart (1969) took the Reynolds number to be small and used Oseen's approximation. For the sphere, the buoyancy causes changes in the velocity which are small relative to the mainstream velocity provided that G is small enough. Consequently, Oseen's approximation is uniformly valid. The case of the hot wire, however, is crucially different. As we shall see, the velocity in the (ideal, steady) wake of a hot wire is infinite at infinite distance from the wire, however small G may be. At some distance from the wire, the change caused by the buoyancy in the velocity in the wake becomes of the same order as the mainstream velocity $\mathbf{u}_{\infty}^{*}$. At this stage, even though the Reynolds number may be small, Oseen's approximation is invalid.

The present treatment centres on the role of the wake. The feebler the buoyancy, the further the buoyant fluid must rise before acquiring a given increase in speed. Thus, for $\epsilon \ll 1$, the length scale for the growth of the wake is large. The flow in the wake, therefore, is largely decoupled from the flow near the wire.

Outside the wake there is a small irrotational flow, induced partly by the inflow at the wake's outer edges and partly by the pressure difference across the wake which is caused by the buoyancy when the wake is inclined to the vertical. The secondary irrotational flow has the same length scale as the wake and so it also is largely decoupled from the flow near the wire. Surprisingly, perhaps, it turns out that the angle between the wake and the mainstream is always small. The reason for this devolves on the pressure differences across the wake. To the irrotational flow, the wake behaves like a vortex sheet. A wake that veered upwards more sharply would imply an irrotational flow in which these pressure differences were larger than is consistent with the forces and accelerations in the wake.

The relative straightness of the inclined wake and the decoupling of the wake and irrotational flow from the flow near the wire are useful simplifications. They permit the wake to be calculated by the usual shear-layer approximations, with the external velocity taken to be that of the mainstream and the wake's width taken to be zero at its base. The only other external factor on which the flow in the wake then depends is the wire's net rate of loss of heat. For the purpose of calculating the wake to a first approximation, the wire's rate of loss of heat may be taken to be the same as in purely forced convection. The irrotational flow may then be readily deduced by treating the wake as a distribution of sinks and vortices along a half line. Moreover, the secondary irrotational flow proves to be responsible for the major part of the change that is caused by buoyancy in the heat lost by the wire. This leads to the main quantitative result of the paper, which is that the change caused by buoyancy in the heat lost by the wire is to a first approximation the same as would be caused in the absence of buoyancy by a certain specified change in the mainstream velocity (see (5.17)).

The approximations used here do not involve a detailed knowledge of the flow near the wire. In particular, an analytic solution for the flow about the wire in the absence of buoyancy is not required. For this reason the Reynolds number does not need to be restricted to be small; and it is restricted only in so far as the flow has been supposed to be steady. The limiting factor here is the stability of the wake, as is noted in §6. The sectional shape of the wire is also largely immaterial to the argument, though it has in fact been taken to be circular. Certain difficulties, which so far are unresolved, arise when the incident stream is nearly horizontal or is slanted downwards. Accordingly, these situations have been excluded.

The plan of the paper is as follows. The wake, the irrotational flow and the diffusive zone round the cylinder are considered in turn in §§3–5. In retrospect, the main quantitative result can be extracted by elementary arguments, concerned largely with orders of magnitude, and this is done in §6.

2. Formulation

We consider the steady two-dimensional flow of an infinite stream of incompressible fluid past a horizontal circular cylinder of diameter d. The incident stream is uniform and has velocity $\mathbf{u}_\infty$ and temperature T_∞ and the cylinder's surface has a uniform temperature T_w, which is greater than T_∞. Heating by viscous dissipation is neglected and the temperature $T_\infty + \theta(T_w - T_\infty)$ at a general point is assumed to vary so slightly that the kinematic viscosity ν, thermal diffusivity κ and thermal coefficient of expansion β are each effectively constant throughout the fluid.

Boussinesq's approximation then applies, and we may write the flux equations in non-dimensional form as

$$\left.\begin{aligned} \mathbf{u}.\nabla\mathbf{u} &= -\nabla p - \epsilon\theta\hat{\mathbf{g}} + \nabla^2\mathbf{u}, \\ \sigma\mathbf{u}.\nabla\theta &= \nabla^2\theta, \\ \nabla.\mathbf{u} &= 0. \end{aligned}\right\} \tag{2.1}$$

Here, $\mathbf{u}$ denotes the non-dimensional velocity, p denotes the non-dimensional change in the pressure from the ambient pressure, $\hat{\mathbf{g}}$ is a unit vector directed vertically downward, $\sigma = \nu/\kappa$ is the Prandtl number and

$$\epsilon = G/R^3, \tag{2.2}$$

where

$$G = \frac{g\beta d^3(T_w - T_\infty)}{\nu^2}, \quad R = \frac{u_\infty^* d}{\nu}. \tag{2.3}$$

The scales that have been adopted to render the lengths, velocities and pressure differences non-dimensional are ν/u_∞^*, u_∞^* and $\rho_\infty u_\infty^{*2}$ respectively, ρ_∞ being the ambient density.

The parameter ϵ provides a measure of the buoyancy, and the basis of the subsequent approximation is that ϵ is small. As an indication of actual magnitudes, it might be noted that $\epsilon \simeq 5/u_\infty^{*3}$ for air when $T_\infty = 20\,^\circ\mathrm{C}$, $T_w - T_\infty = 10\,^\circ\mathrm{C}$ and u_∞^* is in $\mathrm{cm\,s^{-1}}$. Of the remaining parameters, σ will be taken to be of $O(1)$ and R

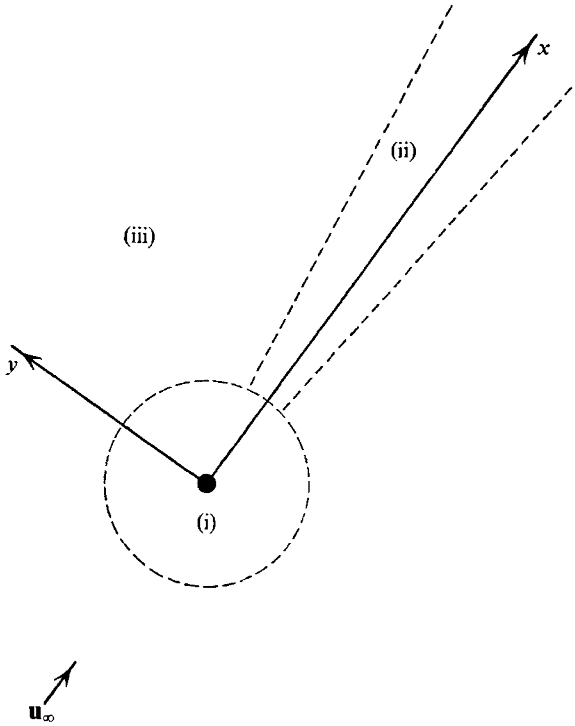

FIGURE 1. The primary zones of the flow: (i) the diffusive zone, where $\rho = O(1)$, (ii) the wake, (iii) the irrotational flow, where $\rho = O((N\epsilon)^{-2})$.

must be less than 44 so that at least the motion is stable in the absence of buoyancy. The question of the instability of the wake is deferred to §6.

As the starting point for our analysis, we shall presume the existence of the following zones:

(i) *a diffusive zone*, at distances of $O(\nu/u_\infty^*)$ from the cylinder, in which diffusion either dominates or balances convection;

(ii) *a wake*, at distances from the cylinder much larger than ν/u_∞^*, in which diffusion balances convection and whose width is much smaller than the radius of curvature of any of its streamlines; and

(iii) *an irrotational zone*, outside both (i) and (ii), in which the vorticity and the temperature difference θ are negligibly small (see figure 1). The wake will also be presumed to convey almost all the heat lost by the cylinder.

We may surmise immediately, from (2.1), that the velocity increment caused by the buoyancy forces of the diffusive zone is small, of $O(\epsilon)$. What is not, perhaps, immediately obvious is that velocity increments of larger order can be generated by an irrotational flow induced by the wake. Since the wake and the irrotational zone are crucial, we shall consider these two first. To this end, we shall use non-dimensional Cartesian co-ordinates Oxy, scaled to ν/u_∞^*, with Ox in the direction of $\mathbf{u}_\infty^*$, Oy inclined upwards (when not horizontal) and the origin O on the cylinder's axis. The corresponding components of $\mathbf{u}$ will be denoted by (u, v) and the corresponding cylindrical polar co-ordinates will be denoted by (ρ, ϕ).

3. The wake

3.1. $\mathbf{u}_\infty$ *vertically upwards*

The simplest situation occurs when the non-dimensional mainstream velocity $\mathbf{u}_\infty = \hat{\mathbf{x}}$ is vertically upward. The wake is then symmetrical about Ox; and, from the outset, we can plausibly assume that the velocity outside the wake differs only slightly from $\mathbf{u}_\infty$.

The wake first forms several viscous lengths (ν/u_∞^*) downstream of the cylinder. For $\epsilon \ll 1$, the buoyancy forces are relatively small everywhere and can cause increases of $O(1)$ in the velocity $\mathbf{u}$ only at large distances from the cylinder. So, at the beginning of the wake, and for some distance thereafter, $\mathbf{u}$ is approximately equal to $\hat{\mathbf{x}}$. Thus, at the beginning of the wake, we have to a first approximation,

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \epsilon\theta + \frac{\partial^2 u}{\partial y^2}, \\ \sigma\frac{\partial \theta}{\partial x} &= \frac{\partial^2 \theta}{\partial y^2}. \end{aligned}\right\} \tag{3.1}$$

In keeping with the notion of a wake, we shall suppose that the temperature difference θ and the velocity difference $(u-1)\hat{\mathbf{x}}$ are much less at the outer edges of the wake than inside the wake. Then (3.1) yields for the temperature and the vertical velocity in the wake

$$\left.\begin{aligned} \theta &= Nx^{-\frac{1}{2}}\theta_0(\eta), \\ u &= 1 - Cx^{-\frac{1}{2}}f(\eta) + N\epsilon x^{\frac{1}{2}}g_1'(\eta), \end{aligned}\right\} \tag{3.2}$$

where

$$\left.\begin{aligned} &\eta = y/\sqrt{x}, \quad \theta_0 = (\pi/\sigma)^{\frac{1}{2}}e^{-\frac{1}{4}\sigma\eta^2}, \quad f = e^{-\frac{1}{4}\eta^2}/2\sqrt{\pi}, \\ &g_1'(\eta) = \frac{2\sqrt{\pi}|\eta|}{1-\sigma}\int_{|\eta|}^{\infty}\left(\frac{e^{-\frac{1}{4}\sigma\eta^2}}{\sqrt{\sigma}} - e^{-\frac{1}{4}\eta^2}\right)\frac{d\eta}{\eta^2}. \end{aligned}\right\} \tag{3.3}$$

The Nusselt number N is here defined as

$$N = -\frac{R}{4\pi}\int_0^{2\pi}\frac{\partial\theta}{\partial\rho}\,d\phi, \tag{3.4}$$

the temperature gradient $\partial\theta/\partial\rho$ being evaluated at the cylinder, where $\rho = \frac{1}{2}R$. The constant C in (3.2) is independent of ϵ and, as in the absence of buoyancy,

$$C = R(\text{drag on the cylinder})/(\rho_\infty u_\infty^{*2}d). \tag{3.5}$$

The solution (3.2) shows the vertical velocity to increase like $N\epsilon x^{\frac{1}{2}}$. The assertion that $\mathbf{u} \simeq \hat{\mathbf{x}}$ is therefore tenable in the wake only for $1 \ll x \ll (N\epsilon)^{-2}$.

As soon as x is of $O((N\epsilon)^{-2})$, and thereafter, the increase in the vertical velocity due to the buoyancy may be of $O(1)$, so that the linear equations (3.1) no longer apply. The flux equations then appropriate are

$$\left.\begin{aligned} \mathbf{u}.\nabla u &= \epsilon\theta + \frac{\partial^2 u}{\partial y^2}, \\ \sigma\mathbf{u}.\nabla\theta &= \frac{\partial^2\theta}{\partial y^2}, \\ \nabla.\mathbf{u} &= 0. \end{aligned}\right\} \tag{3.6}$$

Hence the velocity and the temperature may be determined to $O(1)$, provided that $X = (N\epsilon)^2 x$ is small enough, by

$$\left.\begin{aligned} u &= 1+\sum_{m=1}^{\infty} X^{\frac{1}{2}m} g_m'(\eta), \\ v &= \tfrac{1}{2}x^{-\frac{1}{2}} \sum_{m=1}^{\infty} X^{\frac{1}{2}m}[\eta g_m'(\eta)-(m+1)\,g_m(\eta)], \\ \theta &= Nx^{-\frac{1}{2}} \sum_{m=0}^{\infty} X^{\frac{1}{2}m}\theta_m(\eta). \end{aligned}\right\} \tag{3.7}$$

The profile functions $g_1(\eta)$ and $\theta_0(\eta)$ are given by (3.3) with the added proviso that $g_1(0) = 0$. The remaining profile functions $g_m(\eta)$ and $\theta_m(\eta)$ satisfy the differential equations

$$\left.\begin{aligned} 2g_m''' + \eta g_m'' - m g_m' &= G_{m-1}(\eta) \\ &= -2\theta_{m-1} + \sum_{r=1}^{m-1}[r g_r' g_{m-r}' - (r+1)\,g_r g_{m-r}] \quad (m \geqslant 2), \\ 2\sigma^{-1}\theta_m'' + \eta\theta_m' - (m-1)\,\theta_m &= \Theta_{m-1}(\eta) \\ &= \sum_{r=0}^{m-1}[(r-1)\,g_{m-r}'\theta_r - (r+1)\,g_r\theta_{m-r}] \quad (m \geqslant 1), \end{aligned}\right\} \tag{3.8}$$

with the boundary conditions

$$g_m(0) = 0,\, g_m',\, \theta_{m-1} \to 0 \quad \text{as} \quad \eta \to \pm\infty \quad (m \geqslant 2); \tag{3.9}$$

and these equations admit the unique solutions

$$\left.\begin{aligned} g_m' &= \frac{1}{2}\int_{\infty}^{|\eta|} (|\eta|-\tau)^m e^{-\frac{1}{4}\tau^2}\left\{\int_0^{\tau} e^{\frac{1}{4}t^2}\frac{d^m}{dt^m}G_{m-1}(t)\,dt - B_m\right\}d\tau, \\ \theta_m &= \frac{1}{2}\int_{\infty}^{|\eta|} (|\eta|-\tau)^{m-1} e^{-\frac{1}{4}\sigma\tau^2}\left\{\int_0^{\tau} e^{\frac{1}{4}\sigma t^2}\frac{d^{m-1}}{dt^{m-1}}\Theta_{m-1}(t)\,dt - A_m\right\}d\tau, \end{aligned}\right\} \tag{3.10}$$

where

$$\left.\begin{aligned} A_{2m} &= 2^{m-1}(\sigma/\pi)^{\frac{1}{2}}\int_0^{\infty} e^{-\frac{1}{4}\sigma\eta^2}\left\{\int_0^{\eta} e^{\frac{1}{4}\sigma t^2}\frac{d^{2m-1}}{dt^{2m-1}}\Theta_{2m-1}(t)\,dt\right\}d\eta,\quad A_{2m+1} = 0, \\ B_{2m} &= 0,\quad B_{2m+1} = \frac{2^m}{\sqrt{\pi}}\int_0^{\infty} e^{-\frac{1}{4}\eta^2}\left\{\int_0^{\eta} e^{\frac{1}{4}t^2}\frac{d^{2m+1}}{dt^{2m+1}}G_{2m}(t)\,dt\right\}d\eta, \end{aligned}\right\} \tag{3.11}$$

which define g_m' and θ_m iteratively and are well behaved for large η, giving in fact

$$g_m' = O(\eta^{m-3}e^{-\frac{1}{4}\sigma\eta^2}), \quad \theta_m = O(\eta^m e^{-\frac{1}{4}\sigma\eta^2}) \quad \text{for} \quad \eta \gg 1 \quad \text{and} \quad \sigma < 1. \tag{3.12}$$

For large X the series in (3.7) may cease to be sufficiently accurate, but it suffices for the purpose in hand to note that their basic form,

$$u = u(X,\eta), \quad v = N\epsilon V(X,\eta), \quad \theta = Nx^{-\frac{1}{2}}\tau(X,\eta), \tag{3.13}$$

will be preserved in any further possible continuation of the solution to larger X, as is readily apparent on rescaling the wake equations (3.6).

When X is very large (i.e. for $x \gg (N\epsilon)^{-2}$), there exists an asymptotic solution

$$\left.\begin{aligned} u &= X^{\frac{1}{5}}Q'(z), \quad v = \tfrac{1}{5}N\epsilon X^{-\frac{2}{5}}(2zQ' - 3Q), \\ \theta &= N^2\epsilon X^{-\frac{3}{5}}\,\Gamma(z), \end{aligned}\right\} \tag{3.14}$$

where $$z = N\epsilon y X^{-\frac{2}{5}}(= X^{\frac{1}{10}}\eta).$$

The profile functions Q and Γ in (3.14) are defined by the following differential equations and boundary conditions,

$$\left.\begin{aligned} Q'^2 - 3QQ'' &= 5\Gamma + 5Q''', \\ -3\sigma Q\Gamma &= 5\Gamma', \\ Q', \Gamma \to 0, &\quad z \to \pm\infty, \end{aligned}\right\} \tag{3.15}$$

with the further condition

$$\sigma \int_{-\infty}^{\infty} Q'\Gamma \, dz = 2\pi, \tag{3.16}$$

to ensure that the heat flux from the cylinder equals that across the wake. Both Q and Γ have been evaluated numerically (Yih 1953).

In all, then, the length scale for the action of buoyancy in the vertical wake is $(N\epsilon)^{-2}$; and the velocity difference $\mathbf{u} - \hat{\mathbf{x}}$ is $o(1)$ for $1 \ll x \ll (N\epsilon)^{-2}$, grows to $O(1)$ where $x = O((N\epsilon)^{-2})$ and becomes large like $X^{\frac{1}{5}}$ for $x \gg (N\epsilon)^{-2}$. These stages of the wake's growth will be dubbed, for convenience, the initial, intermediate and asymptotic stages, respectively.

As the fluid in the wake gathers speed it draws in fluid from outside. On this score, we note, for future reference, that the transverse component of the velocity at the edges of the wake may be written as

$$\left.\begin{aligned} v_+ &= N\epsilon V(X, \infty) = N\epsilon F(X), \\ v_- &= N\epsilon V(X, -\infty) = -N\epsilon F(X), \end{aligned}\right\} \tag{3.17}$$

where F is, at most, of $O(1)$. For $X \ll 1$, $F(X)$ may be represented as a series in powers of $X^{\frac{1}{2}}$ and, from (3.1), (3.2) and (3.7),

$$F(0) = -\int_0^{\infty} \theta_0(\eta)\, d\eta = -\pi/\sigma, \tag{3.18}$$

whilst for $X \gg 1$, $F = O(X^{-\frac{2}{5}})$.

3.2. $\mathbf{u}_\infty$ *slanted upwards*

We now take the incident velocity to be slanted upwards at an angle α, where $0 < \alpha < \frac{1}{2}\pi$, to the vertical. In this case, the location of the wake is not *a priori* known. However, as we shall see, the pressure must be approximately constant across the wake and this implies that the wake is, in fact, to a first approximation in line with the incident velocity.

Since the wake is thin relative to the radius of curvature of its streamlines, the wake flow is governed approximately by

$$\left.\begin{aligned} u_s \frac{\partial u_s}{\partial s} + u_n \frac{\partial u_s}{\partial n} &= -\frac{\partial p}{\partial s} + \epsilon\theta \cos\alpha' + \frac{\partial^2 u_s}{\partial n^2}, \\ -\frac{d\alpha'}{ds} u_s^2 &= -\frac{\partial p}{\partial n} + \epsilon\theta \sin\alpha', \\ u_s \frac{\partial \theta}{\partial s} + u_n \frac{\partial \theta}{\partial n} &= \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial n^2}, \\ \frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} &= 0, \end{aligned}\right\} \tag{3.19}$$

where α' denotes the inclination to the vertical of a reference streamline l, chosen arbitrarily from the streamlines of the wake, and s, n denote orthogonal co-ordinates, s being distance along the reference streamline l from some arbitrary point in the diffusive zone and n being distance from l measured along the upwardly sloping normals to l. The change in pressure across the entire width of the wake, $-h \leqslant n \leqslant h$, say, is therefore

$$p(s,h)-p(s,-h) = \epsilon \sin\alpha' \int_{-h}^{h} \theta\, dn + \frac{d\alpha'}{ds}\int_{-h}^{h} u_s^2\, dn. \tag{3.20}$$

The integral $\int_{-h}^{h} \theta\, d\eta$ represents the pressure change associated directly with the buoyancy of the fluid in the wake. Because the wake conveys nearly all the heat lost by the cylinder, we know that

$$\int_{-h}^{h} u_s \theta\, dn \simeq 2\pi N/\sigma. \tag{3.21}$$

We can safely presume that, as for the vertical wake, the streamwise component u_s of the velocity is of $O(1)$, or is larger, and that u_s and θ are positive. So, (3.21) implies that

$$\int_{-h}^{h} \theta\, dn = O(N), \tag{3.22}$$

at most. In other words, the contribution to the pressure change across the wake that is associated directly with the buoyancy of the fluid in the wake is at most of $O(N\epsilon) = o(1)$. The pressure change across the wake also includes the usual contribution associated with the wake's curvature. When u_s is of $O(1)$, this contribution is plainly small, because of the wake's relative thinness. However, u_s may become large through the action of buoyancy, as happens in the vertical wake. In this event, since buoyancy is the primary cause of the acceleration in the wake, we can expect that

$$u_s^2 = O(1) + O\left(\int_0^s \epsilon\theta\, ds\right); \tag{3.23}$$

so that the pressure change associated with the wake's curvature, which is given by

$$\frac{d\alpha'}{ds}\int_{-h}^{h} u_s^2\, dn = \frac{d\alpha'}{ds}\left[O(h) + \int_0^s O(N\epsilon)\, ds\right] = o(1), \tag{3.24}$$

is again small provided that $s(d\alpha'/ds)$ is at most of $O(1)$. The total change in pressure across the wake is thus of $o(1)$.

In order to see what this implies for the irrotational flow outside the wake, we may think of the wake as a vortex sheet and of the diffusive zone as a point singularity. The fact that the pressure change across the wake is small means that, correct to $O(1)$, there is no discontinuity in pressure across the vortex sheet. The complex velocity potential for the irrotational flow, correct to $O(1)$, is therefore regular everywhere, save at most at one point. In addition, the irrotational velocity can be expected to be finite. Hence, correct to $O(1)$, the irrotational velocity must be constant and, as foreshadowed, the wake must be in line with the incident velocity (i.e. $\alpha' \simeq \alpha$).

The flux equations for the slanted wake may now be written as

$$\left.\begin{aligned} u_s\frac{\partial u_s}{\partial s}+u_n\frac{\partial u_s}{\partial n} &= \epsilon\theta\cos\alpha+\frac{\partial^2 u_s}{\partial n^2}, \\ u_s\frac{\partial \theta}{\partial s}+u_n\frac{\partial \theta}{\partial n} &= \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial n^2}, \\ \frac{\partial u_s}{\partial s}+\frac{\partial u_n}{\partial n} &= 0. \end{aligned}\right\} \tag{3.25}$$

As for the vertical wake, we suppose that $u_s - 1$ and θ are much less at the outer edges of the wake than they are inside the wake. The streamline l enters into the equations (3.25) and their boundary conditions only in so far as it is the base-line for the co-ordinates (s, n). So to different choices of l there correspond solutions with different spatial distributions of velocity. This non-uniqueness may be seen another way. If we introduce a stream function defined by

$$\psi = \int_0^n u_s ds \tag{3.26}$$

and re-write the flux equations, with s and ψ as variables, in the form

$$\left.\begin{aligned} u_s\frac{\partial u_s}{\partial s} &= \epsilon\theta\cos\alpha+u_s\frac{\partial}{\partial\psi}\left(u_s\frac{\partial u_s}{\partial\psi}\right), \\ \sigma\frac{\partial u_s}{\partial s} &= \frac{\partial}{\partial\psi}\left(u_s\frac{\partial\theta}{\partial\psi}\right), \end{aligned}\right\} \tag{3.27}$$

then it is clear that, if $\quad u_s = I(s,\psi), \quad \theta = J(s,\psi) \qquad (3.28)$

defines a solution then so also does

$$u_s = I(s,\psi-\psi_0), \quad \theta = J(s,\psi-\psi_0), \tag{3.29}$$

where ψ_0 is a constant. Apart from this arbitrariness in the reference streamline, however, the flux equations (3.25) and their boundary conditions correspond precisely with the equations and boundary conditions for the vertical wake. Thus, the solutions for the slanted wake may be obtained from those for the vertical wake by putting $\epsilon\cos\alpha$ in place of ϵ, and s, n, u_s and u_n respectively in place of x, y, u and v. The reference streamline is thereby selected to be the central streamline, relative to which the temperature and streamwise component of velocity are even functions of n. The precise location of this streamline remains to be determined.

Like the vertical wake, the slanted wake grows through a linear stage, where $1 \ll s \ll (N\epsilon\cos\alpha)^{-2}$ and $\mathbf{u} \simeq \mathbf{u}_\infty$, an intermediate stage, where $s = O((N\epsilon\cos\alpha)^{-2})$ and $\mathbf{u}-\mathbf{u}_\infty = O(1)$, and an asymptotic stage, where $s \gg (N\epsilon\cos\alpha)^{-2}$ and $|\mathbf{u}| \gg 1$. Also, as before, the slanted wake draws fluid from outside, and the transverse component of velocity at the edges of the wake can be represented by

$$\left.\begin{aligned} (u_n)_+ &= N\epsilon\cos\alpha\, V(S,\infty) = N\epsilon\cos\alpha F(S), \\ (u_n)_- &= N\epsilon\cos\alpha\, V(S,-\infty) = -N\epsilon\cos\alpha F(S), \end{aligned}\right\} \tag{3.30}$$

where $F(S)$ is the function introduced in (3.17) and $S = (N\epsilon\cos\alpha)^2 s$.

However, whereas the vertical wake induces a secondary irrotational flow solely by virtue of the inflow at its edges, the slanted wake induces a secondary flow partly by virtue of the inflow at its edges and partly by virtue of the small change in pressure that occurs across the wake. So, for the slanted wake, this change in pressure also needs to be specified. The transverse change in pressure was considered earlier in order to show that $\alpha' - \alpha$ is small and that the flow in the irrotational flow is nearly uniform. We are now preparing for the determination of the small departures from this uniform flow. From the behaviour of u and θ for the vertical wake, we know how u_s and θ behave in the slanted wake in each of the three stages of the wake's growth. Thence on devising suitable functions of S as bases for comparison, we have that in the slanted wake

$$u_s^2 = O(1+S^{\frac{2}{5}}), \quad \epsilon s\theta = O\left(\frac{S^{\frac{1}{2}}(1+S^{\frac{3}{5}})}{1+S^{\frac{1}{2}}}\right) \quad \text{for all} \quad s \gg 1. \tag{3.31}$$

Thus, on returning to (3.20), we see that the pressure change associated directly with the buoyancy of the fluid in the wake is much greater than that associated with the curvature of the streamlines, provided that

$$s\frac{d\alpha'}{ds} = o(S^{\frac{1}{2}}/(1+S^{\frac{1}{2}})) \quad \text{for all} \quad s \gg 1. \tag{3.32}$$

This condition is clearly satisfied when S is not small, because

$$s\frac{d\alpha'}{ds} = O(\alpha' - \alpha) = o(1).$$

For the moment, we shall assume that this condition also holds when S is small. Then, to a first approximation,

$$2\Delta p = p(s,h) - p(s,-h) \simeq \epsilon \sin\alpha \int_{-h}^{h} \theta \, dn \tag{3.33}$$

and, from the expressions (3.2), (3.7), (3.13) and (3.14) for θ, we have

$$\Delta p \simeq N\epsilon \sin\alpha \int_0^\infty \tau(S,\eta)\, d\eta = N\epsilon \sin\alpha G(S), \tag{3.34}$$

where $G(S)$, like $F(S)$, may be represented as a series in $S^{\frac{1}{2}}$, for small S, with leading term

$$G(0) = \int_0^\infty \theta_0 \, d\eta = \pi/\sigma, \tag{3.35}$$

and $G(S) = O(S^{-\frac{1}{5}})$ for large S. The validity of (3.32) for small S will be verified later after the angle $\alpha' - \alpha$ between the wake and the incident velocity has been determined.

Before proceeding, it is worth adding that the above discussion is unlikely to extend readily to the case of a *downwardly* forced draught. To begin with, the buoyancy would probably bring the flow to rest at a distance from the cylinder of $O((N\epsilon\cos\alpha)^{-2})$ and cause reversed flow farther from the cylinder. If this were to happen, the wake approximation (3.19) might well prove to be inadequate. Further, no similarity solution like (3.14), in which the velocity *increases* with distance from the cylinder, seems likely to apply. For if, on the one hand, the velocity in such a solution were directed towards the cylinder, then large upward velocities would be assigned to fluid which had not yet reached the source of the buoyancy, namely the cylinder. If, on the other hand, the velocity in such a solution were directed away from the cylinder, then the fluid involved would be required to accelerate downwards, against the action of buoyancy.

4. The irrotational zone

We now turn to the secondary flow induced outside the wake by the net influx $(u_n)_- - (u_n)_+ = -2(u_n)_+$ at its two edges and by the change $2\Delta p$ in the pressure across its width. Outside the diffusive zone, this flow is effectively irrotational and is slow relative to u_∞ as has already been shown. Accordingly, the total head in the irrotational zone must be nearly uniform and the pressure change $2\Delta p$ must correspond to a change across the wake of $-2\Delta p$ in u_s. Since the wake is thin, the velocity at each edge of the wake may be assigned to the irrotational flow on the wake line $n = 0$, so that, for $S \gg (N\epsilon)^2$,

$$\left.\begin{aligned} u_s(S, 0_+) - u_s(S, 0_-) &= -2\Delta p = -2N\epsilon \sin\alpha G(S), \\ u_n(S, 0_+) - u_n(S, 0_-) &= 2(u_n)_+ = 2N\epsilon\cos\alpha F(S). \end{aligned}\right\} \tag{4.1}$$

We see from (4.1) that the secondary irrotational flow has velocities of $O(N\epsilon)$ and a length scale of $O((N\epsilon)^{-2})$. In order to determine the irrotational flow approximately, the diffusive zone can be treated as a single point because of its relatively smaller extent. Moreover, because the wake line is near to the leeward axis Ox, as defined in § 2 for incident velocities that are not necessarily vertically upward, the boundary conditions (4.1) can be transferred to Ox. Thence, to a first approximation,

$$u = \bar{u} \mp N\epsilon \sin\alpha\, G(X), \quad v = \alpha - \alpha' \pm N\epsilon\cos\alpha F(X) \quad \text{for} \quad Y = 0\pm, \quad (X > 0), \tag{4.2}$$

where
$$X = (N\epsilon\cos\alpha)^2 x, \quad Y = (N\epsilon\cos\alpha)^2 y \tag{4.3}$$

and the mean value $\bar{u}(X)$ is at present undetermined. The secondary irrotational flow can now be represented by a distribution of sinks and vortices along the half line $Y = 0, X > 0$. Hence we find that the secondary irrotational flow is given to a first approximation by

$$u - iv = 1 + \frac{N\epsilon}{\pi}\int_0^\infty \frac{F(X')\cos\alpha - iG(X')\sin\alpha}{X + iY - X'}\,dX', \tag{4.4}$$

where $F(X)$ and $G(X)$ are determined, as noted in § 3, by the flow in the wake.

The mean of the values of v at $Y = \pm 0$ determines the inclination of the wake line $n = 0$ to the incident velocity. Thus, to $O(N\epsilon)$ and for $X \gg (N\epsilon)^2$,

$$\alpha' - \alpha = \frac{N\epsilon\sin\alpha}{\pi}⨍_0^\infty \frac{G(X')}{X' - X}\,dX'. \tag{4.5}$$

In particular, for $1 \gg X \gg (N\epsilon)^2$, we have, with the aid of (3.35), that

$$\alpha - \alpha' \simeq N\epsilon\sin\alpha(\log X)/\sigma. \tag{4.6}$$

Thence, $X(d\alpha'/dX) = o(X^{\frac{1}{2}})$, for small X, as was required (in an equivalent form) by (3.32) in order to justify taking the pressure difference across the wake to be directly due to buoyancy.

When $Z = X+iY$ is large,

$$u-iv = \begin{cases} 1+O(N\epsilon\sin\alpha Z^{-\frac{1}{2}}) & (\alpha \neq 0), \\ 1+O(N\epsilon Z^{-\frac{3}{2}}) & (\alpha = 0). \end{cases} \tag{4.7}$$

More pertinently, when Z is small,

$$\begin{aligned} u-iv &= 1-\frac{N\epsilon}{\sigma}e^{i\alpha}\log(-Z)+N\epsilon c+O(N\epsilon Z^{\frac{1}{2}}) \\ &= 1+N\epsilon c-\frac{2N\epsilon}{\sigma}e^{i\alpha}\log(N\epsilon\cos\alpha) \\ &\quad -\frac{N\epsilon}{\sigma}e^{i\alpha}\log(-x-iy)+o(N\epsilon) \\ &= 1+u_0-iv_0-\frac{N\epsilon}{\sigma}e^{i\alpha}\log(-x-iy)+o(N\epsilon), \end{aligned} \tag{4.8}$$

where

$$\left.\begin{aligned} c &= \frac{1}{\pi}\left\{\int_0^1 \{i[G(s)-G(0)]\sin\alpha-[F(s)-F(0)]\cos\alpha\}\frac{ds}{s}\right. \\ &\qquad\qquad \left.+\int_1^\infty \{iG(s)\sin\alpha-F(s)\cos\alpha\}\frac{ds}{s}\right\}, \\ u_0-iv_0 &= N\epsilon\left(c-\frac{2e^{i\alpha}}{\sigma}\log(N\epsilon\cos\alpha)\right). \end{aligned}\right\} \tag{4.9}$$

The $\log Z$ appears in (4.8) because the influx into the wake and the pressure difference across the wake are both initially non-zero (see (3.18), (3.30) and (3.35)). Thus the diffusive zone, where $x+iy = O(1)$, is effectively immersed in an outer flow, whose complex velocity comprises a uniform contribution $1+u_0-iv_0$ together with a contribution varying as $\log(-x-iy)$. The effect of this outer flow on the convection in the diffusive zone will be dealt with in the next section.

First, however, let us consider the deflexions of the irrotational velocity and of the wake produced by the buoyancy. From the first line of (4.8) it is clear that the irrotational velocity for very small Z is less steeply inclined to the horizontal than the incident velocity. Likewise, as is shown by (4.6), the wake is initially less steeply inclined. These deflexions due to the buoyancy are away from the upward vertical. Deflexions, due to buoyancy, towards the upward vertical are more familiar; so there is seemingly a paradox here which calls for explanation. The crux of the matter is that the dynamics of the wake preclude the transverse buoyancy force from producing a significant transverse acceleration there. The transverse buoyancy force therefore has to be countered by an opposing pressure gradient, and it is to this *counter force* that the irrotational flow responds. That is to say, in order to counter buoyancy, the pressure p on the upper side of the wake has to be greater than on the lower side, and, in consequence, the oncoming fluid (in the irrotational zone) is deflected to the lower side of the wake. The wake's initial deflexion $\alpha'-\alpha$ away from the upward vertical is consistent with this trend.

5. Velocity at the periphery of the diffusive zone

It remains to assess the effect of buoyancy on the diffusive zone. When the Reynolds number is not small, the buoyancy forces within distances ρ of $O(1)$ from the cylinder induce changes in velocity of $O(\epsilon)$, as is clear from (2.1). When R is small, the temperature difference θ is of $O(1/\log R)$ (Wood 1968; Hieber & Gebhart 1968) at distances of $O(1)$ from the cylinder (i.e. in the Oseen zone) and the buoyancy forces acting within these distances induce changes in velocity of $O(\epsilon/\log R)$, the corresponding changes due to buoyancy forces acting within distances of $O(R)$ from the cylinder (i.e. in the Stokes zone) being relatively smaller because of the Stokes zone's relatively smaller width. In either case, the velocities accruing from the buoyancy forces acting at distances up to $O(1)$ from the cylinder may be typified as being of $O(N\epsilon)$.

The velocity in the diffusive zone is not only affected by the local buoyancy. It is also affected by the secondary flow created by the wake. Further, the irrotational secondary flow was found to have velocities which were in the main of $O(N\epsilon)$. So the indirect effect of the buoyancy in the wake on the flow in the diffusive zone is likely to be at least as important as that of the buoyancy in the diffusive zone itself; and, in fact, it proves to be more important, when ϵ is small enough.

To assess the influence of the irrotational flow on the diffusive zone, we consider the outer part of the diffusive zone, where $1 \ll \rho \ll (N\epsilon)^{-2}$, and we use the Oseen approximation. It is convenient at this point to designate a particular radius ρ_1 as the inner boundary of the peripheral zone. The radius ρ_1 is taken to be large but independent of ϵ. Then for $\rho_1 < \rho \ll (N\epsilon)^{-2}$, we assume that

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial x} &= -\nabla p - \epsilon\theta\hat{\mathbf{g}} + \nabla^2\mathbf{u}, \\ \sigma\frac{\partial \theta}{\partial x} &= \nabla^2\theta, \\ \nabla.\mathbf{u} &= 0. \end{aligned}\right\} \tag{5.1}$$

These equations, of course, admit the isothermal solutions

$$\left.\begin{aligned} \mathbf{u}_f' &= u_0\hat{\mathbf{x}} + v_0\hat{\mathbf{y}} + \chi\hat{\mathbf{x}} + \nabla(\Phi-\chi), \\ p_f' &= -\frac{\partial\Phi}{\partial x}, \\ \theta_f' &= 0, \end{aligned}\right\} \tag{5.2}$$

where

$$\chi = e^{\frac{1}{2}x}\sum_{n=0}^{\infty} A_n K_n(\rho/2)\cos n\phi, \quad \Phi = B_0\log\rho + \sum_{n=1}^{\infty} B_n\rho^{-n}\cos n\phi,$$

and A_n, B_n are constants, subject to the conditions

$$\sum_{n=0}^{\infty} A_n = -C'/2\pi, \quad B_0 = \sum_{n=0}^{\infty} A_n = C'/2\pi, \tag{5.3}$$

imposed by the momentum and continuity integrals. Here C' is a drag coefficient, scaled to the appropriate drag in the same way as C was scaled to the drag in

(3.5). The drag relevant to C' becomes apparent when the various contributions to the velocity in the diffusive zone are pieced together. These solutions are independent of the buoyancy forces in the diffusive zone. The uniform part (u_0, v_0) of the buoyancy-induced irrotational velocity has been incorporated in (5.2) in anticipation of the matching of the diffusive zone with the irrotational zone. For large ρ, then, $\mathbf{u}'_f = u_0\hat{\mathbf{x}} + v_0\hat{\mathbf{y}} + O(\rho^{-1})$, save that, where $y = O(x^{\frac{1}{2}})$, $x > 0$,

$$\begin{aligned} u'_f &= u_0 + \chi + O(x^{-1}) \\ &= u_0 + \sum_{n=0}^{\infty} A_n \left(\frac{\pi}{x}\right)^{\frac{1}{2}} e^{\frac{1}{2}(x-\rho)} + O(x^{-1}) \\ &= u_0 - \frac{C'}{2(\pi x)^{\frac{1}{2}}} e^{-\frac{1}{4}\eta^2} + O(x^{-1}), \end{aligned} \tag{5.4}$$

in conformity with the usual asymptotic solution for the wake of a cylinder in a uniform stream.

Our main concern, however, is with the additional velocity and pressure due to the buoyancy forces in the diffusive zone. From (5.1) it follows that

$$\nabla^2 p = \epsilon\left(\frac{\partial\theta}{\partial x}\cos\alpha + \frac{\partial\theta}{\partial y}\sin\alpha\right), \tag{5.5}$$

so, using the Oseen equation for θ, we have as a particular solution for p,

$$p'_b = \epsilon\sigma^{-1}\left[\{\theta + N\log\rho\}\cos\alpha + \left\{\int_{-\infty}^{x}\frac{\partial}{\partial y}(\theta + N\log\rho)\,dx\right\}\sin\alpha\right]. \tag{5.6}$$

This solution is valid for $\rho_1 < \rho \ll (N\epsilon)^{-2}$ save possibly in the strip $|y| \leqslant \rho_1$, $\rho > \rho_1$, $x > 0$, which lies downstream of the inner circle $\rho = \rho_1$ and which will be called the shadow zone. The solution does not necessarily apply in the shadow zone, because for points in the shadow zone the integration in (5.6) includes points in the inner part, $\frac{1}{2}R < \rho \leqslant \rho_1$, of the diffusive zone, where Oseen's approximation may not apply. Since the width of the wake increases, at first as $x^{\frac{1}{2}}$, the wake is much wider than the shadow zone at a distance ρ_2, say, from the cylinder, where $\rho_1^3 \leqslant \rho_2 \ll (N\epsilon)^{-2}$. At this stage, the heat lost by the wire is carried almost entirely by the wake, so only a relatively small proportion of the heat flux from the wire is conveyed across the width of the shadow zone (i.e. across the line $x = \rho_2$, $-\rho_1 < y < \rho_1$). Hence almost all the heat lost by the wire must cross one or other of the two lines running in the upstream direction from (ρ_2, ρ_1) and $(\rho_2, -\rho_1)$ to $(-\infty, \rho_1)$ and $(-\infty, -\rho_1)$ respectively. In the Oseen approximation, $\int_{-\infty}^{x} (\partial\theta/\partial y)\,dx$ represents the combined heat flux across the line of integration due to conduction and convection. So the value of this integral must change by approximately $2\pi N$ as (x, y) is displaced from the point (ρ_2, ρ_1) on one side of the shadow zone to the point $(\rho_2, -\rho_1)$ on the other. This confers a change in p'_b of $O(N\epsilon)$ on crossing the shadow zone at $x = \rho_2$. However, p'_b is $O(N\epsilon)$, with respect to $N\epsilon$. There is no evidence to suggest that p can increase with x faster than $\log x$ (for $x \gg 1$) or that the length scale for significant lateral changes in p can be less than $x^{\frac{1}{2}}$ (for $x \gg 1$). Hence the change in p

on crossing the wake at $x = \rho_2$ would be expected to be *at most* of $O(N\epsilon \log \rho_2)$, and the change in p on crossing the relatively narrower shadow zone at $x = \rho_2$ would be expected to be at most of $O(N\epsilon \rho_1 \log \rho_2/\surd\rho_2)$, which is much less than the change of $O(N\epsilon)$ obtained above. This anomaly is removed by the $N \log \rho$ added to θ in the integral in (5.6) because of the compensating change of approximately -2π that occurs in the value of $\int_{-\infty}^{x} \partial/\partial y\,(\log \rho)\,dx$ when x, y crosses the shadow zone from (ρ_2, ρ_1) to $(\rho_2, -\rho_1)$, and it is for this reason that the second $\log \rho$ in (5.6) has been included. The reason for including the first $\log \rho$ in (5.6) relates to the transverse forces in the shadow zone. From (5.1) and (5.6) we have for the points $(\rho_2, \pm\rho_1)$ on either side of the shadow zone that

$$\left[v_b' - \frac{\partial v_b'}{\partial x} + \frac{\partial u_b'}{\partial y}\right]_{-\rho_2}^{\rho_2} = \int_{-\rho_2}^{\rho_2} \left[-\frac{\partial p_b'}{\partial y} + \epsilon \sin\alpha\theta\right] dx$$
$$= -\frac{\epsilon\cos\alpha}{\sigma}\int_{-\rho_2}^{\rho_2} \frac{\partial}{\partial y}(\theta + N\log\rho)\,dx + \epsilon\frac{\sin\alpha}{\sigma}[\theta + N\log\rho]_{-\rho_2}^{\rho_2}, \tag{5.7}$$

where $\mathbf{u}_b' = (u_b', v_b')$ is a particular solution for the buoyancy-induced velocity consistent with the particular solution p_b' for the buoyancy-induced pressure. Both $u-1$ and $\epsilon\theta$ are of $O(N\epsilon)$ with respect to $N\epsilon$, and, as before, there is no evidence to suggest that $u-1$ increases faster with x than $x^{\frac{1}{2}}$ or that either of v or θ increases faster with x than $\log x$, whilst the length scale for transverse changes in either $\mathbf{u}$ or θ is unlikely to be smaller for large x than the $x^{\frac{1}{2}}$ which is characteristic of the wake. Thus, the difference between the values of the integral in (5.7) at $y = \rho_1$ and at $y = -\rho_1$, on either side of the shadow zone, can be at most $O(N\epsilon\rho_1(\log\rho_2)/\surd\rho_2) = o(N\epsilon)$. Hence, it is again necessary to annex $N\log\rho$ to θ, because, essentially as explained above, the difference between the values of $\int_{-\rho_2}^{\rho_2} \partial\theta/\partial y\,dx$ for $y = \rho_1$ and for $y = -\rho_1$ is of $O(N\epsilon)$.

After substituting p_b' for p in the momentum equations and exploiting their similarity to the heat equation, we obtain, as a particular solution for the buoyancy-induced velocity,

$$\left.\begin{aligned} u_b' &= \epsilon\left[\frac{1}{1-\sigma}\int_{-\infty}^{x}(\theta - \zeta)\,dx + U_1\right]\cos\alpha + \epsilon U_2 \sin\alpha,\\ v_b' &= \epsilon(U_2\cos\alpha - U_1\sin\alpha), \end{aligned}\right\} \tag{5.8}$$

where

$$\left.\begin{aligned} U_1 &= \frac{\sigma\zeta - \theta}{\sigma(1-\sigma)} - \frac{N}{\sigma}\log\rho, \quad U_2 = \int_{-\infty}^{x}\frac{\partial U_1}{\partial y}\,dx,\\ \zeta &= Ne^{\frac{1}{2}x}K_0(\tfrac{1}{2}\rho). \end{aligned}\right\} \tag{5.9}$$

The variable ζ satisfies

$$\partial\zeta/\partial x = \nabla^2\zeta, \tag{5.10}$$

and has been included in U_1 to ensure that U_2 does not change by $O(N\epsilon)$ across the shadow zone, the net flux of ζ $\left(\text{viz. } \int_0^{2\pi}(\partial\zeta/\partial\rho - \zeta\cos\phi)\,\rho\,d\phi\right)$ from the source at $\rho = 0$ being equal to $-2\pi N$. These formulae again apply in the part of the peripheral zone that is outside the shadow zone.

To round off the specification of the buoyancy-induced velocity we note, from (5.1), that θ can be represented by

$$\theta = e^{\frac{1}{2}\sigma x} \sum_{n=0}^{\infty} C_n K_n(\tfrac{1}{2}\sigma\rho) \cos n\phi, \tag{5.11}$$

where the C_n are constants restricted, so as to preserve the heat flux by

$$\sum_{n=0}^{\infty} C_n = N. \tag{5.12}$$

Since all the derivatives of θ may safely be taken to be periodic in ϕ for $\rho = \rho_1$, we may presume that $C_n K_n(\sigma\rho_1/2) = o(n^{-q})$ for large n and for any constant $q > 0$. Hence, it may be shown that, for large ρ,

$$\theta = e^{-|O(\rho)|},$$

except that where also $y^2 = O(x)$, $x > 0$,

$$\theta \sim \left(\frac{\pi}{\sigma x}\right)^{\frac{1}{2}} N e^{-\frac{1}{4}\sigma\eta^2},$$

which accords with the (linear) wake approximation for $x \ll (N\epsilon\cos\alpha)^{-2}$ given previously.

We can now see that, for large enough ρ, the buoyancy-induced velocity outside the wake is given by

$$u'_b - iv'_b \sim \epsilon e^{i\alpha}(U_1 - iU_2) \sim -\frac{N\epsilon}{\sigma} e^{i\alpha} \log(-\rho e^{i\phi}), \tag{5.13}$$

the parts of $\mathbf{u}'_b$ directly dependent on ζ and θ being small like $e^{-|O(\rho)|}$. Thus, buoyancy in the viscous zone does in fact produce at its periphery a logarithmically increasing velocity that matches the $\log(-x-iy)$ component obtained in (4.8) for the irrotational zone. Since the total change $\mathbf{u}_b$ in the velocity due to the buoyancy is small throughout the entire diffusive zone, it is governed there by linear equations, say

$$L[\mathbf{u}_b] = 0, \tag{5.14}$$

which reduce to (5.1) if R is small. So, we can write

$$\mathbf{u}_b = \mathbf{u}''_b + \mathbf{u}''_f, \tag{5.15}$$

where $\mathbf{u}''_b$ is a particular solution of (5.14) that for large ρ equals to a first approximation the particular solution $\mathbf{u}'_b$ found above for the outer periphery of the diffusive zone and $\mathbf{u}''_f$ is a solution of (5.14) with buoyancy omitted. For large ρ, $\mathbf{u}'_f$ has, to a first approximation, the form of the appropriate peripheral solution $\mathbf{u}''_f$. For definiteness, both $\mathbf{u}''_b$ and $\mathbf{u}''_f$ will be taken to vanish at the cylinder. In order to complete the matching of $\mathbf{u}_b$ in the diffusive zone with the irrotational velocity just outside the diffusive zone, we require that

$$(u''_f, v''_f) \to (u_0, v_0) \quad \text{as} \quad \rho \to \infty. \tag{5.16}$$

Since this limiting velocity is of $O(N\epsilon \log (N\epsilon))$, the forced convective velocity $\mathbf{u}_f''$ associated with it will also be of $O(N\epsilon \log (N\epsilon))$. The velocity $\mathbf{u}_b''$ due to the buoyant forces in the diffusive zone is, however, of $O(N\epsilon)$, for the reasons given at the beginning of this section. Hence the forced convective velocity $\mathbf{u}_f''$ is dominant. Consequently, the effect of buoyancy in the diffusive zone, where $\rho = O(1)$, is to a first approximation the same as would result from an increase in the velocity at infinity by an amount

$$u - iv = -\frac{2N\epsilon}{\sigma} \log (N\epsilon \cos \alpha)\, e^{i\alpha} \tag{5.17}$$

in the absence of buoyancy.

6. Summary and discussion

Apart from the important matter of why the wake is nearly in line with the incident velocity, for which the reader is referred to §3, the gist of the argument so far is as follows.

When the buoyancy is slight enough, the change it causes in the velocity is small in the diffusive zone and only grows to mainstream magnitude in the wake at a relatively large distance from the diffusive zone.

The diffusive zone, defined as the zone where $\rho = O(1)$ and diffusion either dominates or balances convection, is thus of the same extent as in the absence of buoyancy. The change in velocity in the diffusive zone caused solely by the buoyancy force there is readily seen to be of $O(N\epsilon)$, which is taken to be small.

Where the wake speed is $O(1)$, the wake widens as $x^{\frac{1}{2}}$, x being distance from the wire. So, since the wire's heat loss is almost entirely conveyed through the wake, the non-dimensional temperature must decrease as $Nx^{-\frac{1}{2}}$. The streamwise buoyancy-induced acceleration in the wake is therefore of $O(N\epsilon x^{-\frac{1}{2}})$ and, over distances of $O(x)$, causes changes in the streamwise speed of $O(N\epsilon x^{\frac{1}{2}})$. Thus these changes in speed in the wake become comparable to the mainstream speed at distances of order $(N\epsilon)^{-2} \gg 1$ from the wire.

Beyond the diffusive zone and outside the wake, the vorticity is negligible and the temperature is effectively ambient. But a secondary irrotational motion is induced by the wake, in two ways. First, as the fluid in the wake gathers speed, it draws fluid from the outside. Second, because the wake fluid is relatively light, the dynamic pressure beneath the wake is less than that above the wake.

The magnitude of the net inflow $-2(u_n)_+$ to the wake is closely related to the wire's heat loss, since, with the integrals taken across the wake,

$$-2(u_n)_+ = \int \frac{\partial u_s}{\partial s}\, dn = 0\left(\frac{1}{u_s}\int u_s \frac{\partial u_s}{\partial s}\, dn\right) = O\left(\frac{\cos\alpha}{u_s}\int \epsilon\theta\, dn\right) = O\left(\frac{\cos\alpha}{u_s^2}\int \epsilon u_s \theta\, dn\right)$$
$$= O(2\pi N\epsilon \cos\alpha/(\sigma u_s^2)). \tag{6.1}$$

Thus the inflow is $O(N\epsilon)$ where u_s is $O(1)$. The pressure increase $2\Delta p$ across the wake is solely due to buoyancy, i.e.

$$\Delta p = \sin\alpha \int \epsilon\theta\, dn, \tag{6.2}$$

because, whilst the longitudinal buoyancy force ($O(\epsilon\theta)$) balances the longitudinal acceleration and viscous force, the transverse buoyancy force ($O(\epsilon\theta)$) dominates its relatively smaller transverse counterparts. So, as above,

$$\Delta p = O(\pi N\epsilon \sin\alpha/(\sigma u_s)) \quad (\ll 1) \tag{6.3}$$

and is $O(N\epsilon)$ where u_s is $O(1)$. Most importantly, at the periphery of the viscous zone, where the wake forms, the convection velocity $\mathbf{u} \simeq \hat{\mathbf{x}}$, and the order of magnitude statements of (6.1) and (6.3) can be replaced by approximate equalities (on putting $u_s = 1$ but leaving $\partial u_s/\partial s$), giving

$$(u_n)_+ \simeq -(\pi N\epsilon\cos\alpha)/\sigma, \quad \Delta p \simeq (\pi N\epsilon \sin\alpha)/\sigma. \tag{6.4}$$

To evaluate the irrotational flow approximately, we replace the wake by a line, one of its streamlines say, at which the changes $2\Delta p$ in pressure and $-2(u_n)_+$ in transverse velocity that occur across the wake are taken to occur discontinuously. Since Δp and $(u_n)_+$ are both small, of $O(N\epsilon)$, we affirm that the secondary motion is small and that the wake is only slightly inclined to the mainstream velocity $\hat{\mathbf{x}}$. The wake speed, and hence also Δp and $(u_n)_+$, are therefore, to first order, independent of the secondary flow, and they have the longitudinal length scale $(N\epsilon)^{-2}$ imposed by the buoyancy. The secondary flow caused by Δp and $(u_n)_+$ can be expected to have the same typical length. So, since the width of the diffusive zone is much less than $(N\epsilon)^{-2}$, the line on which the discontinuities are assigned can be taken, to a first approximation, to extend along the axis Ox downstream of the wire's centre, the values of Δp and $(u_n)_+$ at the periphery of the diffusive zone being taken to apply at $x = 0$.

The secondary velocity caused by the discontinuities in p and v of $O(N\epsilon)$ along Ox is plainly, overall, of $O(N\epsilon)$. But since the discontinuities at $x = 0$, which may be taken to be given by (6.4), are non-zero, a logarithmic singularity occurs. Thus, bearing in mind that the length scale for Δp and $(u_n)_+$ is $(N\epsilon)^{-2}$, we have directly that the complex secondary irrotational velocity, calculated with the approximations mentioned above, reduces to

$$u - iv = -\frac{N\epsilon}{\sigma} e^{i\alpha} \log[-(x+iy)(N\epsilon)^2] + O(N\epsilon) \quad \text{for} \quad x \ll (N\epsilon)^{-2}. \tag{6.5}$$

This represents the velocity at the outer periphery of the diffusive zone and outside the wake.

Thus, the diffusive zone, because of the secondary flow induced by the wake, is effectively embedded in a velocity slightly different from that of the mainstream. The largest uniform component of this outer (complex) velocity is

$$-2\sigma^{-1}N\epsilon \log(N\epsilon)\, e^{i\alpha}$$

and this produces changes in velocity in the diffusive zone, indirectly due to buoyancy, of $O(N\epsilon \log N\epsilon)$. On the other hand, the change in velocity in the diffusive zone due to the buoyancy forces in the diffusive zone is of $O(N\epsilon)$, as noted above. (Examination of the Oseen equations with buoyancy included, which hold in the outer part of the diffusive zone, confirms that the buoyancy forces in the diffusive zone give rise to a complex velocity $u_b'' - iv_b''$ increasing like

$N\epsilon\sigma^{-1}\log -(x+iy)$ for large $x+iy$ and, in this way, accounts for the largest non-uniform components of the secondary flow just outside the diffusive zone.) Hence the dominant effect of buoyancy is independent of the buoyancy forces in the diffusive zone and is equivalent to a change of mainstream velocity, with buoyancy absent, of amount $-2\sigma^{-1}N\epsilon\log N\epsilon$ at an angle 2α to the vertical, i.e. to a change of $-2\sigma^{-1}N\epsilon\log(N\epsilon)\cos\alpha$ in the mainstream speed and to a change of $\Delta\alpha = -2\sigma^{-1}N\epsilon\log(N\epsilon)\sin\alpha$ in the angle of the mainstream to the vertical. Because the component of $\hat{\mathbf{g}}$ along the wake is $\hat{\mathbf{g}}\cos\alpha$ the length scale for the development of the buoyancy-induced velocities of $O(1)$ is $(N\epsilon\cos\alpha)^{-2}$ rather than $(N\epsilon)^{-2}$. It is for this reason that $\log(N\epsilon\cos\alpha)$ appears in most of the statements elsewhere in the paper as to the effective change in the mainstream velocity. No extra accuracy can be claimed, however, for using $\log(N\epsilon\cos\alpha)$ in place of $\log N\epsilon$, inasmuch as the case where $\cos\alpha$ is small has been excluded.

For the application to hot-wire anemometers in air, it is important to note the severe restriction on the smallness of $N\epsilon$ for which the assumption made here of steady, two-dimensional flow with a uniform velocity at infinity is reliable. The region which, through buoyancy, affects a wire's heat loss has been shown to have a width of order

$$L = \nu/u_\infty(N\epsilon\cos\alpha)^2 = u_\infty^5/N^2g^2\beta^2(T_w-T_\infty)^2\nu\cos^2\alpha. \tag{6.6}$$

So, as soon as the flow has significant three-dimensional variations or the ambient stream is significantly non-uniform over lengths of this order, the treatment given above needs reassessing. If we take, as sample figures, $\Delta T = 30\,^\circ\mathrm{C}$, $d = 5\times10^{-3}$ cm, $\nu = 0{\cdot}15\ \mathrm{cm^2\,s^{-1}}$, $\alpha = 0$ and speeds of 3 and 4 $\mathrm{cm\,s^{-1}}$, we have roughly the results of table 1.

	N	$N\epsilon$	L
$u_\infty = 3\ \mathrm{cm\,s^{-1}}$	0·22	0·12	3·6 cm
$u_\infty = 4\ \mathrm{cm\,s^{-1}}$	0·23	0·053	14 cm

TABLE 1

These are appreciable lengths in the context of laboratory experiments. So particularly for the smaller values of $N\epsilon$ (say less than 0·05), the *prima facie* possibility exists that buoyancy effects are inseparably tangled with those of three-dimensionality due to the wire's supports or with non-uniformity in the flow to be measured.

Another extraneous factor to be reckoned with is the onset of turbulence in the wake. The Reynolds number based on wake width and wake velocity is of $O((N\epsilon\cos\alpha)^{-1})$ at distances of $O(L)$ from the wire, i.e. at the stage where the velocity induced by buoyancy is of $O(u_\infty)$. If turbulence occurs closer to the wire the change in heat loss due to buoyancy is likely to be appreciably altered but otherwise if turbulence is deferred to much further from the wire (i.e. to a distance $\gg L$) the change in heat loss due to buoyancy is likely to be little affected. Unfortunately, there appears to be no available experimental or theoretical determination for the critical Reynolds number, the nearest relevant evidence being some measurements for a point source of heat in free convection in which the corresponding Reynolds number based on wake width at transition was of the order of 10^2 (Yih 1953).

REFERENCES

Collis, D. C. & Williams, M. J. 1959 *J. Fluid Mech.* **6**, 357.

Gebhart, B. & Pera, L. 1971 *J. Fluid Mech.* **45**, 49.

Hieber, C. A. & Gebhart, B. 1968 *J. Fluid Mech.* **32**, 21.

Hieber, C. A. & Gebhart, B. 1969 *J. Fluid Mech.* **38**, 137.

Wood, W. W. 1968 *J. Fluid Mech.* **32**, 9.

Yih, C-S. 1953 Fluid models in geophysics. *Proc. 1st Symp. on the Use of Models in Geophysical Fluid Dynamics*, Johns Hopkins (ed. R. R. Long).